17 Solving the problem

Experienced problem solvers like your lecturers often seem to look at a problem and just know how to solve it as if by magic. Going from the statement of a problem to a chosen method of solution is one of the things students find hardest in engineering, science and technology courses. Nobody can give you a foolproof method that solves every problem every time but a combination of experience, gained from tackling lots of problems, and a systematic approach to planning your strategy will take you a long way.

In this chapter you will:

1. see how to analyse a problem initially
2. get an overview of strategies for solution
3. investigate the benefits of working in symbols
4. consider ways to tell if your solution is working.

USING THIS CHAPTER

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Solving the problem

Estimate your current levels of confidence. At the end of the chapter you will have the chance to reassess these levels where you can incorporate this into your personal development planner (PDP). Mark between 1 (poor) and 5 (good) for the following:

Getting started on a problem.	Choosing a method for solution.	Working with symbols.	Knowing if you are heading the right way.

Date: ________________

1 The problem-solving process

In this chapter we shall look at how you go from a statement of a problem to a solution. Problems come in all different shapes and sizes and the problems you meet during your studies will of course be very dependent on your subject. Nobody can give you a magic method that you can just apply without thinking to get the answer to any problem, but you can develop a systematic approach to problems that makes the process easier to manage. You are aiming to produce a toolkit of methods, not a recipe book of solutions, so taking time to understand the theory on which your problem is based is crucial (see Chapter 16, Section 2).

Learning a to use a process works just the same way as learning new facts; proficiency comes with practice and experience, so you need to take the solution methods described in this chapter and get in the habit of applying them each time you have a problem to solve. Improving your techniques and strategies for solving maths-based problems will help you to gain better marks and to learn more effectively and efficiently.

The problem-solving process can be divided into two parts: **thinking** and **doing**. Within each part you can systematically carry out a number of different tasks for each problem you try to solve.

Thinking and doing

For successful problem solving you need to:

1. **Understand the problem** by working out what information you have been given in the problem statement and what other information you will need.
2. **Devise a strategy** for solving the problem.

This is the ***thinking*** part of the process.

Then you need to:

3. **Carry out the strategy** for solving the problem.
4. **Write down the steps** for the solution.
5. **Check your answer**.

This is the ***doing*** part of the process.

Have a look at Figure 17.1 which summarises the steps to take in each part of the process.

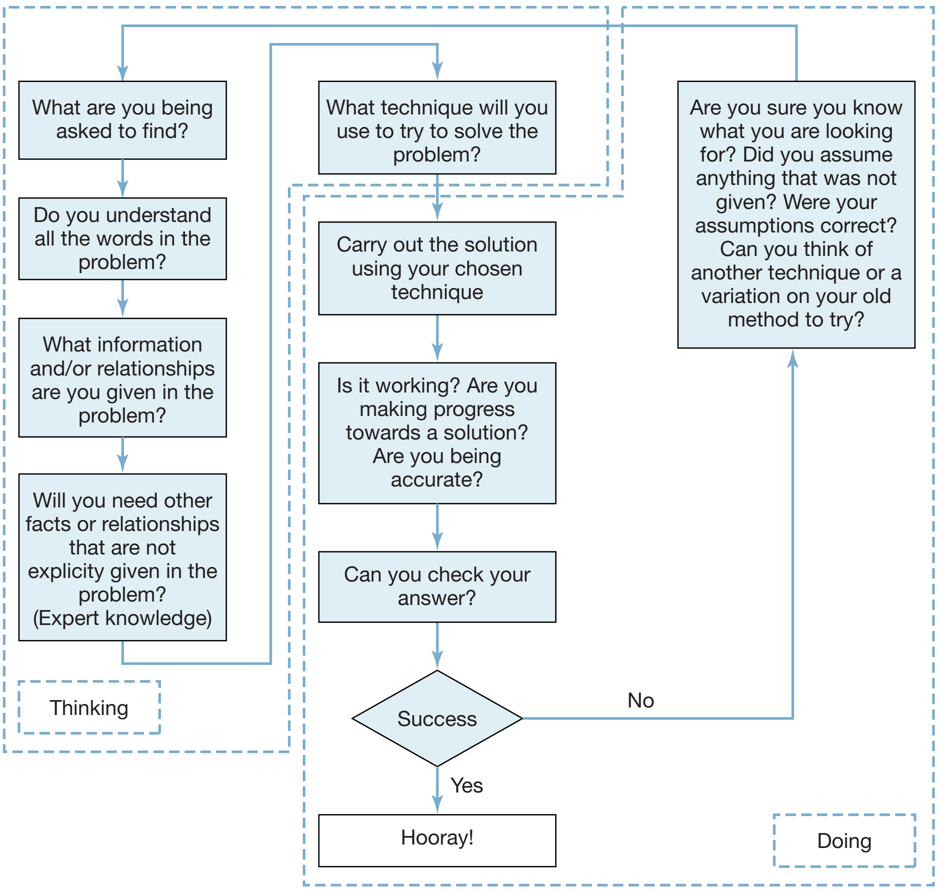


Figure 17.1 Steps in the problem-solving process

As you can see from the figure, the process contains a loop that you only leave when the problem is solved. One of the biggest problems for the novice problem solver is having the stamina to go round the loop again and again until the problem is solved and the experience to try a slightly different method each time. The other big problem for the beginner is knowing when to give up and get help. The methods described here can help with these judgements, but again there is no substitute for practice and experience. It’s a bit like running a marathon: you can read about the best tactics, but the only way to achieve your goal is by actually going out and running. In the same way that you would gradually increase your training distances, you need to increase slowly the complexity of the problems you tackle.

2 Understanding the problem

It is never wise to start by trying to apply a solution method. First you need think about your problem and really try to understand what you are dealing with. This might seem unnecessary with problems where you can see how to do them straight away, but you need to train your brain on these easier questions to approach the harder ones the right way. Get into good habits early and you will have an easier time as the difficulty of the problems you meet increases.

Closed and open problems

We can think of two broad classes of problems:

- **Closed problems**: where all the data you need to get to a solution is in the problem statement. You may have to come up with a method or choose which physical laws to apply but you don’t expect to have to make estimates of, for example, the numerical values to use for different quantities. These are the kinds of problems you normally meet in a traditional exam.
- **Open problems**: where you may have to research some information or where some of the facts or values you need may not be known at all. These are more like the problems that researchers try to solve. You might come across this kind of problem in lab work or in an extended project.

ACTIVITY 1 Closed or open?

Are the following problems closed or open?

	Closed or open?
Two dustcarts collect bags of rubbish. One collected 47 bags and one collected 56 bags. How many bags of rubbish were collected all together?	Closed
What would be the best type of dam to use to dam the River Thames?	Open
Solve $2x + 3y = 1$ $x - 4y = 17$	
Estimate the distance from the earth to the sun.	
If Tom has three times as many apples as Susan and Susan has a quarter as many as Joe, who has four, how many does Mary have if Mary has two more than Tom?	
A round wheel with a radius of 37 cm rolls at a constant speed of 3 revolutions per second. How far does the axle of the wheel move in 8 seconds?	
Is it better for you to walk or cycle to university?	

See the feedback section.

What are you trying to achieve?

Whether the problem is closed or open, the first step towards solution is to **define what you are trying to find**. In some problems you will find a sentence that makes this clear. In other problems it is harder to work out what is needed, and in some very complicated problems, working out what will count as a valid solution is a whole problem in itself. Make sure you really understand your problem. This is the first part of the thinking stage.

You need:

- First, to make sure you know what all the words in the problem mean. If you find one you don't understand then look it up in a dictionary. Is it an everyday word or a specialist or technical word? (See 'Cracking the code' in Chapter 16.)
- Next, to identify the required result. You might like to underline it or rewrite it in your own words. What will it look like? Will it be a number; a conclusion in words, a proof, a diagram or something else? (For some further hints on how to identify parts of a question see the BUG technique in Chapter 12.)

For more open problems, the first attempt to define what you are looking for may be very vague, but thinking about why it is vague can be a valuable way to start to make the problem definition more precise.

You must always keep your goal in mind as you work through a problem.

Hot Tip

If you are looking for something, you will never find it if you don't know what it is!

Example: Identifying what you are trying to achieve

In the problem below, the part of the statement that tells you about the solution has been underlined.

> Two pickup trucks collected bags of rubbish. One collected 56 bags of rubbish and the other collected 47 bags of rubbish. <u>How many bags of rubbish were collected all together?</u>

The solution you are looking for will be a number.

ACTIVITY 2 What are you looking for?

What about the problems below? Simply identify what you are trying to find and what form the solution will take.

	The solution will be …
If Tom has three times as many apples as Susan and Susan has a quarter as many as Joe, who has four, how many does Mary have if Mary has two more than Tom?	
Design an efficient tin opener.	
Solve: $2x + 3y = 1$ $x - 4y = 17$	

See the feedback section.

Hot Tip

If you can't get started on a problem at all, the most likely reason is that you haven't properly identified what you are looking for. Once you know what you are after, it is easier to see ways you might try to get to it.

What are you given?

The next step in solving a problem is to work out what information you have been given in the problem statement.

You can write a list or underline parts of the question as you prefer. (See the BUG technique in Chapter 12.)

You need to be very clear about what you do and don't know. It is very easy to make assumptions without really knowing you have made them. It may help to get in the habit of writing down what you are given in your own words.

Example: Identifying what you have been given

Let's return to the problem we considered before:

> Two pickup trucks collected bags of rubbish. One collected 56 bags of rubbish and the other collected 47 bags of rubbish. How many bags of rubbish were collected all together?

You are given two pieces of information in the question:

Truck 1 collected 56 bags
Truck 2 collected 47 bags

In this simple example you may feel that what you have been given is obvious, but it is worth getting into the habit of asking yourself this question explicitly even for simple problems. Once it is second nature, you will be able to use the same framework to deal with more complicated problems.

ACTIVITY 3 Identifying givens

For the problems below, identify the information you have been given to help you solve the problem:

Solving the problem

	I have been given …
If Tom has three times as many apples as Susan and Susan has a quarter as many as Joe, who has four, how many does Mary have if Mary has two more than Tom?	
Design an efficient tin opener.	
Solve $2x + 3y = 1$ $x - 4y = 17$	

See the feedback section.

What else do you know?

The information you are given in a problem statement is often not the only information you need to solve the problem. Frequently you are assumed to know something else as well. This is especially true in open problems, but even in closed problems you are not always told which method to use and must decide for yourself.

The extra knowledge may be common sense or well-known information or it may be specialist knowledge you have obtained from studying your course. All these additional pieces of knowledge you bring to a problem are part of your own **expert knowledge**.

This can be summarised by the diagram in Figure 17.2.

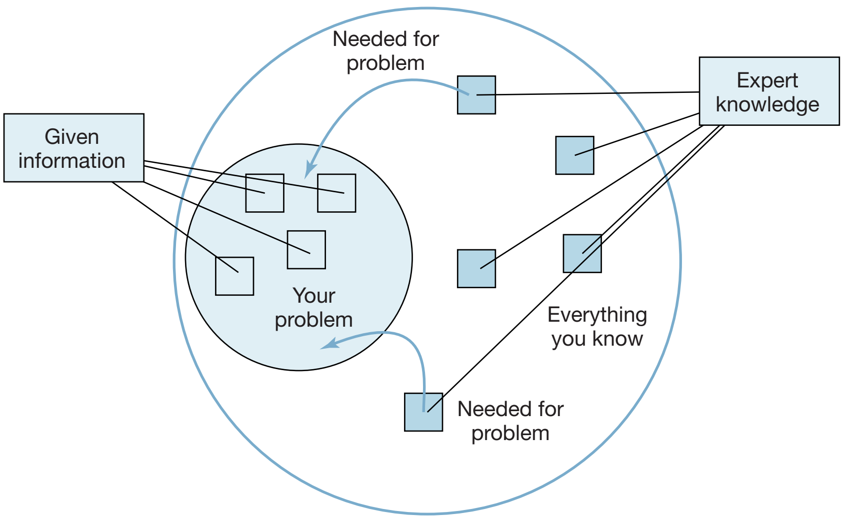


Figure 17.2 Using your expert knowledge

Expert knowledge might be a physical law, a piece of common knowledge, a piece of specialist language or something you have learnt from experience. For example, in the problem below we can see that there is some common knowledge needed and also some understanding of simple technical terms.

Problem	Expert knowledge needed
A round wheel with a radius of 37 cm rolls at a constant speed of 3 revolutions per second. How far does the axle of the wheel move in 8 seconds?	The writer of this problem is assuming you know what is meant by the common terms *round*, *wheel*, *rolls* and also by the more technical terms *radius*, *revolutions per second* and axle. You also need to know a physical law: the relationship between the radius of a circle and its circumference. None of this information is given to you by the problem statement.

ACTIVITY 4 Using your expert knowledge

What assumptions is the problem setter making about what you know in these next two problems?

Problem	Expert knowledge needed
1. Mary looked out of her window and saw some chickens and cows passing by. She counted all the legs of the chickens and cows and found that the total number of legs added up to 66. How many of each kind of animal passed by her window if the total number of animals is 24?	
2. A car travels at 60 mph. How far does it travel in 30 minutes?	

See the feedback section.

Hot Tip

Don't forget to make a clear note of the **expert knowledge** you decide you may need as this will help you to decide on a strategy for solution.

3 Strategies for solution

Once you have really analysed your problem thoroughly you will know what you are looking for and what you know already. You will also have identified the expert knowledge you think you might need. Now you are ready to think about solution strategies.

In routine problems (see Section 5), which test memory and your ability to apply basic skills, you may be told in the problem statement which method to use.

Examples

1. By using Simpson's rule with six intervals, determine

$$\int_0^{\frac{\pi}{2}} \sqrt{(2{\cdot}5 - 2{\cdot}5\cos 2\theta)}\,d\theta$$

2. Put the following equations into matrix $Ax = b$:

$2x + 3y = 1$
$x - 4y = 17$

then find the inverse matrix A^{-1} and hence solve the set of equations for x and y.

You may need to learn or revise the methods, but there is no mystery here about what you are expected to do: use Simpson's rule in the first example and use matrix methods in the second.

In more complex problems, you are more often left to work out what method to use for yourself.

Examples

1. Integrate $\int \frac{\sin\theta}{\cos\theta} d\theta$

2. If 150 m of fencing is required to enclose three sides of a rectangular field, while the remaining side is bordered by a river, determine the lengths of the sides such that the area enclosed will be a maximum.

In the first example you know you need to use integration, but you will need to bring your expert knowledge gained from lectures and textbooks into play to decide which of the various methods of integration will be the right one here to get you to the answer. In the second example there is more than one possible method. If you have studied differentiation the word 'maximum' might start you thinking in a profitable way; if not, you might try a geometric method perhaps or a drawing.

Generally, students find selecting a method for solution one of the hardest skills to develop, especially when the problem is expressed in words rather than in mathematical symbols. In Section 2 you have already thought about how to analyse a problem when you first meet it, and this certainly helps you to make a start. Look at all the things you know about the problem and all the expert knowledge you feel might be related to it. Can you think of strategies you might try? If you can think of more than one strategy for your problem, write yourself a list so you don't forget your other ideas while you try out your first one.

Once you have an idea about how to start, you need to be very systematic. The process you need is shown in Figure 17.3.

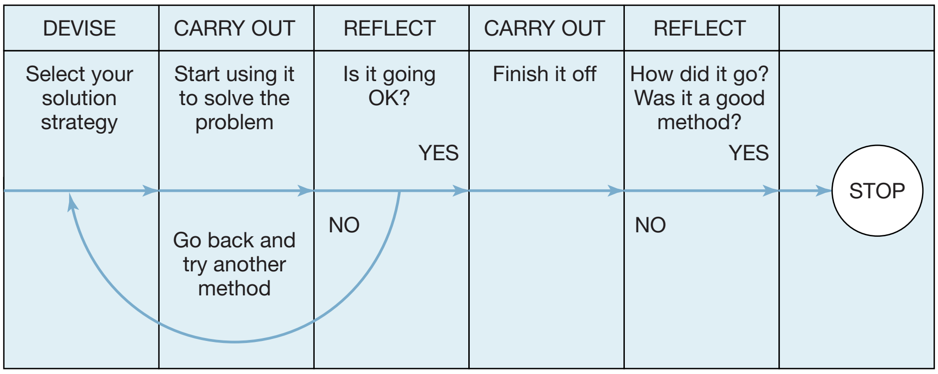


Figure 17.3 The problem-solving process

> **Hot Tip**
>
> Remember: you have only failed to solve the problem when you have run out of strategies to try!

But what if you can't begin to think of a strategy at all?

George Pólya was a mathematician, born in Budapest in 1887, who noticed that his students often had trouble solving problems even though they knew a lot of maths. To help them he tried to write a methodology for problem solving. In the end he found that, while there were no hard and fast rules, there were a number of strategies that all worked well at least some of the time. He couldn't find a rule to tell you which strategy to use in any particular case, but he said that all the methods listed below (and some others) were worth considering (Pólya, 1990). You just pick the one that looks most promising for your particular problem and give it a try.

As your experience grows you should be aiming to develop a toolkit of solution methods based on a deep understanding of how the underlying maths works, not a recipe book where you just follow a set of steps to get the answer. This level of skill takes time and effort to develop but by doing all the examples you are set you are going in the right direction.

Making a picture

This tactic is so fundamental to problem solving in science, engineering and technology it should be second nature to you; as soon as you see a problem statement you start to think how you might represent it as a picture. Drawing a diagram helps you to analyse the problem and can sometimes lead directly to a solution. Diagrams, graphs and drawings for problem solving don't need to be beautiful works of art, but they do need to be informative to you, the user.

You can draw any diagram that helps *you* to understand the problem you are considering. It may be a sketch of equipment, a map, a graph or bar chart, a pie chart, a calendar, a Venn diagram, a picture, a spider diagram, a mind map or something else.

Examples

1. A metal block of mass 100 kg rests on a plane surface inclined at $\theta = 20°$ to the horizontal. The coefficient of friction, μ is 0.75. A force is applied to the block with a rope parallel to the slope of the plane. What is the smallest force required to move the block up the plane? The acceleration due to gravity may be approximated as 9.8 kg m s^{-2}.

Problem analysis

What are you trying to find?

The force required to pull the block up the plane.
The answer will be a number of newtons.

What are you given?

Mass of block $m = 100$ N
Angle of block to horizontal $\theta = 20°$
Coefficient of friction $\mu = 0.75$

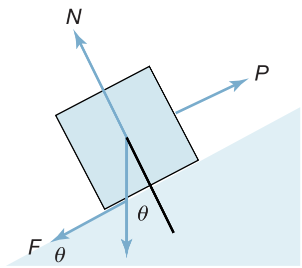
What expert knowledge might you need? (This will come from your lectures or from reading a mechanics textbook.)

- The weight of the block $W = mg$, which acts vertically downwards.
- When the force is just enough for the block to start to move the system is in limiting equilibrium. The sum of forces acting parallel to

the plane is zero. The sum of the forces acting perpendicular to the plane is zero.

- The friction force, *F*, acts in the opposite direction to the direction of motion.
- A normal reaction, *N*, acts perpendicular to the plane.
- *F* and *N* are related by the relation $F = \mu N$, where μ is the coefficient of friction.

At this point a diagram with the forces indicated by arrows should help you to visualise the situation.



Your diagram may look something like this where *W* is the weight of the block and *P* is the force pulling up the slope. Note the reminder of the relationship between *F* and *N* jotted down next to the diagram.

ACTIVITY 5 Drawing a diagram

What kind of diagram would you draw to help with each of the problems below?

Problem	Diagram
1. John is going to cook a big meal for his friends. He wants to serve the main course at 12:30 and the desert approximately 30 minutes later. He will cook roast lamb, which takes 10 minutes to prepare, 2 hours to cook and 15 minutes to rest and carve. Also, roast potatoes, which take 10 minutes to prepare and 90 minutes to cook; sprouts, which take 8 minutes to prepare and 20 minutes to cook; and gravy, which takes 10 minutes to prepare and is finished with the contents of the roasting tin after the lamb is removed. For desert he will make apple pie, which takes 30 minutes to prepare and 40 minutes to cook. He needs to find time to whip some cream (10 minutes) to serve it with. There can be a gap between preparing an item and cooking it. Can you help John to plan how to get the food to the table?	

Problem	Diagram
2. A uniform ladder, of length L, rests against a smooth wall at an angle of θ to the floor. The coefficient of friction between the ladder and the floor is mS. Show that the shallowest angle at which the ladder is stable is given by the equation $\theta = \tan^{-1}\left(\frac{1}{2\mu_s}\right)$	.
3. The professor invited all 40 of the students on his course to tea and provided them with jelly, ice cream and cake. The students who arrived early were hungry and soon all the food was gone. The students who arrived later were faced with an empty table! Fourteen students ate cake, 13 ate jelly and 16 ate ice cream. Three of them ate jelly and cake. Five had cake and ice cream. Eight had jelly and ice cream and two had a plate with all three desserts on. How many students got nothing to eat?	

See the feedback section.

Breaking the problem into parts

Many problems are easier to solve if you break them down into smaller parts, solve each part separately and then build a final answer from the solutions to the smaller parts. This works especially well for more complex problems. Breaking the problem down into parts makes it more manageable. Often you can solve the easier parts first and worry about the harder questions later. This helps you to build your confidence about the problem.

ACTIVITY 6 Breaking into parts

1. You wish to build a dam across a river. How will you design your dam?

The solution to this problem is made up from the solutions to a whole set of related problems. First, consider some quite **broad questions** like those shown in the table below. Can you think of some more that might apply?

Broad initial questions
What is the budget?
What is the time scale for construction?

These then lead to more **specific questions** like those below. You can't answer these until you have answered at least some of the first set of questions Can you think of more of these questions?

More specific questions
Where will you put the dam?
What shape should it be?

You may have added questions about materials to use, number of workers required, equipment needed, etc.

Now pick one of your more specific questions from above and try to break it down further into detailed questions you will need to answer.

Detailed questions

2. Can you break this problem down into the steps you need to carry out to get to a solution?

> A cubic polynomial is given as $P(x) = ax^3 + 7x^2 + 2x - b$. When $x = -1$, $P(x) = 0$ and when $x = 1$, $P(x) = 8$. Sketch the graph for $P(x)$ showing clearly where it crosses both axes.
>
> Step 1:
> Step 2:
> Step 3:
> Step 4:
> Step 5:
>
> etc.

HINT: You will need to find the roots of the polynomial. See the feedback section.

Solving a simpler, related problem

Sometimes we are faced with a problem that looks too complicated to solve straight away. We can often make some progress by solving similar, but simpler problems until we can see a pattern emerging. Choosing a related problem that gives you insight into the original problem is something you can only learn with practice. Whenever you can use this method, however, it is likely to save you a substantial amount of time and also give you a great feeling of satisfaction.

Example

Find the value of

$$\frac{2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20 + 22 + 24 + 26 + 28 + 30 + 32 + 34 + 36 + 38}{3 + 6 + 9 + 12 + 15 + 18 + 21 + 24 + 27 + 30 + 33 + 36 + 39 + 42 + 45 + 48 + 51 + 54 + 57}$$

Why do you think this problem looks hard? Probably because there are a lot of numbers on the top and bottom of the fraction. It would be easy to make a mistake when you added them all up.

The most obvious way to simplify it is to reduce the size of the problem.

Have you spotted a pattern yet? Do you need to try some more to convince yourself?

$\frac{2}{3}$?	
$\frac{2+4}{3+6}$?	
$\frac{2+4+6}{3+6+9}$?	

So what do you guess the solution to the first problem to be?

ACTIVITY 7 Simplify to find the pattern

1. The factors of 360 add up to 1170. What is the sum of the reciprocals of the factors?

HINT: Expert knowledge required: you need to know what a factor is and what a reciprocal is before you can make any progress. Don't forget to make sure you know what you are trying to find.

2. Find the sum of the first 25 odd numbers.

Getting a feel for the size of the answer

Often, when you can't see the answer to a problem straight away it helps to try to estimate the approximate size of the answer. For example, if you calculate the distance of the earth from the sun an get an answer of 22 km then you can be pretty sure your answer is wrong! If you make a point of thinking how big or small an answer might be before you start, then errors in order of magnitude are easier to spot and you can avoid mistakes.

ACTIVITY 8 How big and how small?

Estimate the approximate magnitude of the following:

Item	Approximate size
Wavelength of ultraviolet light	
Land area of Great Britain	
Thickness of a grain of salt	
Diamter of the earth	
Diameter of a golf ball	
Speed of an average person walking	
Distance from the earth to the sun	
Diameter of a human hair	

See the feedback section.

As well as being able to estimate realistic values for physical quantities it is useful to be able to estimate the answer to arithmetical problems. Again, if you estimate before you calculate you have a check on whether your answer is reasonable.

ACTIVITY 9 How big will the answer be?

Try to estimate an answer for these calculations without using a calculator. You are not trying to get the exact answer, just to get an idea of the size of number your answer should be. Instead of using the exact numbers you can use numbers that are easier to deal with to construct an estimator. The numbers you choose for your estimator need to be of the same size as the original ones. Using scientific notation can simplify the process further and often shows you where parts of the calculation cancel each other out. The first two rows give you examples.

Problem	Estimator	Approximate answer
983 × 256 × 198	1000 × 250 × 200 $= 1 \times 10^3 \times 25 \times 10 \times 2 \times 100$	50×10^6
(2439 × 1220 × 56) ÷ 430	(2500 × 1200 × 50) ? 400 = (25 × 100 × 12 × 100 × 5 × 10) ÷ (4 × 100) = (25 × 3 × 100 × 5 × 10) ÷ (1)	375×10^3
5236 ÷ 7805 + 2300 − 4534		
(34 387 × 54 567) + 585 734		
((385 730 × 348) – 57) ÷ 680		

See the feedback section.

You can find out more about these problem-solving strategies and others that Pólya suggested by reading his book *How to solve it* or by looking at the book by Posamentier and Krulik (1998) *Problem Solving Strategies for Efficient and Elegant Solutions*, which gives lots of examples of how to apply Pólya's techniques to both mathematical and more everyday problems.

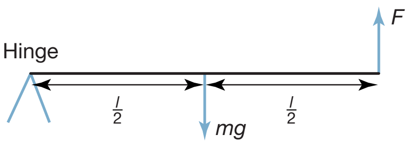
Above and beyond anything else, you need to practise problem solving using your structured approach. This is the best way to develop your skills. Remember, your lecturers probably aren't cleverer than you; they just have more problem-solving experience.

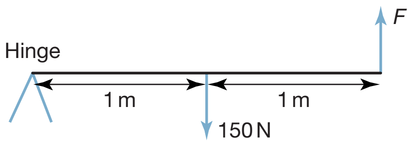
4 Putting the numbers in

In many problems you have a choice of whether to work with symbols to begin with and to put numbers in later or whether to work with numbers from the start.

Which of the two solutions below might be most useful when you come to revise the topic?

A hinged trapdoor of mass 15 kg and length 1 m is to be opened by applying a force F at 90° to the door surface at the opposite end to the hinge. Calculate the magnitude of the force F. Assume the acceleration due to gravity is 10 ms^{-2}.

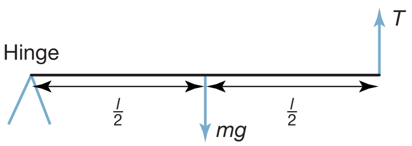
<table>
<tr><th>Using symbols first:</th><th>Using numbers first:</th></tr>
<tr><td>
Let the mass of the door be m. Then the weight of the door is mg where g is the acceleration due to gravity and let l be the length of the beam.


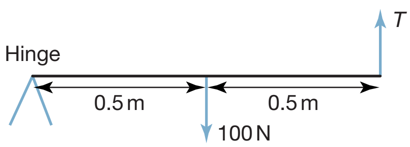
Taking moments about the hinge,
$mg \times \frac{1}{2} - F \times l = 0$ (a)
Rearranging (1) gives,
$F = \frac{mg}{2}$
Then substituting for m and g,
$F = \frac{15 \times 10}{2} = 75\,\text{N}$
</td><td>
If mass of door is 15 kg, weight of door can be taken as 150 N.


Taking moments about the hinge,
$150 \times 0.5 - F = 0$
so
$75 - F = 0$
$F = 75\,\text{N}$
</td></tr>
</table>

Now look at the problem below and compare it with the problem above. You should be able to see the similarities even if you haven't studied this topic before.

A uniform beam of length 2 m is attached to a wall at one end by a hinge. The mass of the beam is 10 kg. The beam is supported at the other end by a rope which is attached to the ceiling vertically above the end of the beam. Find the tension in the rope. Assume the acceleration due to gravity is 10 m s^{-2}

<table>
<tr><th>Using symbols first:</th><th>Using numbers first:</th></tr>
<tr><td>
Let the mass of the beam be m, then the weight of the beam is mg. Let g be the acceleration due to gravity and let l be the length of the beam.


</td><td>
If mass of beam is 10 kg, weight of beam can be taken as 100 N.


</td></tr>
</table>

Using symbols first:	Using numbers first:
Taking moments about the hinge, $mg \times \frac{l}{2} - T \times l = 0$ (1) Rearranging (1) gives $T = \frac{mg}{2}$ Then substituting for m and g, $T = \frac{10 \times 10}{2} = 50\,\text{N}$	Taking moments about the hinge, $100 - T \times 2 = 0$ so $100 - 2T = 0$ $T = 50\,\text{N}$

Using the symbols until close to the end **highlights the similarities** between the two problems and allows you to see patterns easily. The more patterns you can spot, the less you need to memorise methods. You can see that it is often much harder to spot the similarities and patterns when you put the numbers in early in the solution process.

5 Carrying out your strategy

At this stage of your solution you should have worked out

- what you are trying to find,
- what you have been told,
- what else might be useful,
- one or more ideas about how you could proceed.

You are now ready to try out a solution.

If you are lucky, the first method you try will work. If not, you need to know when to stop trying one way and to try another. Is it the wrong method or have you made a mistake along the way?

Knowing whether your solution is working

You may start out with one or more solution methods in mind. You pick the most likely one and start your solution. How do you know if it is working? At what point should you abandon it and try another way?

Unfortunately there is no hard and fast rule about this. Just like the ability to think of a variety of methods in the first place, knowing if your method is going to work comes with experience. However, there are some questions that you may ask yourself that can help:

Useful questions to check your progress	Comments
Are you confident that each step is correct as you take it?	See Chapter 18 for advice about spotting errors as you work.
Can you imagine what the next step will be? And the next? And the one after that?	You should be looking forward after each step. If you are heading in the right direction, things should be becoming clearer. You should be more aware after each step how you are going to proceed to your goal. The looking forward is like playing a strategy game. The best chess players know all the options for many moves ahead; a novice knows some moves for the next step. You can develop this skill with practice.
Have you used *all* the information you were given?	In textbook problems you are usually given no extraneous information, so if you have a use for it all, it's probably a good sign. In real-life problems this may not be true, so use this question with care.

6 On reflection

This chapter has taken you through the steps in solving a problem. First you really need to get to grips with what you are looking for and what information you have been given. Only then can you start to think about how you might solve your problem. Solving the problem itself is often an iterative process: you try a solution to see if it works and, if not, you adapt it a bit or try a different method until you get there. With plenty of practice you can build up your abilities and develop a systematic approach that gives you a sturdy framework for problem solving.

Summary of this chapter

Have look at Figure 17.4 and make sure you are familiar with the concepts.

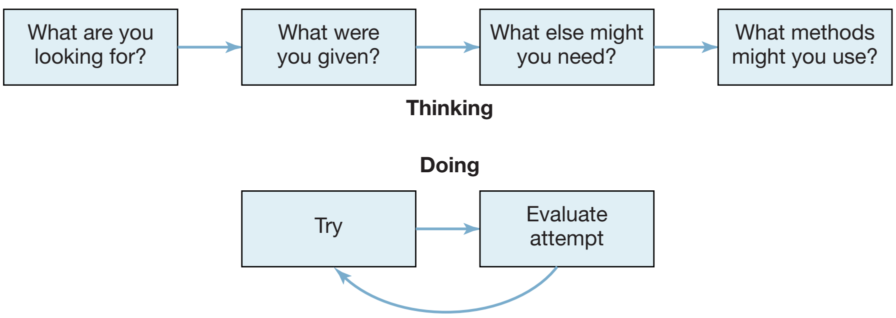


Figure 17.4 Summary of the problem-solving process

ACTIVITY 10 Update your personal development planner

Reflect on your current abilities and consider what needs to improve. You may want to transfer this information to your institution's personal development planner scheme.

Grade your confidence on a scale of 1–5 where 1 = poor and 5 = good.

My developing skills	Confidence level 1–5	Plans to improve
Getting started on a problem.		
Choosing a method for solution.		
Working with symbols.		
Knowing if you are heading the right way.		

Date: ________________

Getting extra help

- Is there a skills centre or drop-in maths workshop you can attend?
- Is there a timetabled workshop or tutorial provided to help answer your questions?
- Does the tutor who set the problem sheet have 'office hours' where you can go and ask for help?

Feedback on activities

ACTIVITY 1 Closed or open?

	Closed or open?
Two dustcarts collect bags of rubbish. One collected 47 bags and one collected 56 bags. How many bags of rubbish were collected all together?	**Closed** – you need to come up with a method but no further data.
What would be the best type of dam to use to dam the River Thames?	**Open** – you need to know where it is to be dammed, what the dam is for, how wide the river is, etc.

	Closed or open?
Solve $2x + 3y = 1$ $x - 4y = 17$	**Closed** – you need to know a method to solve simultaneous equations but all the facts about *x* and *y* and their relationship are contained in the question.
Estimate the distance from the earth to the sun.	**Open** – you can just have a guess or you can try to improve your estimate by researching facts, but there is no information at all to help you in the question statement.
If Tom has three times as many apples as Susan and Susan has a quarter as many as Joe, who has four, how many does Mary have if Mary has two more than Tom?	**Closed** – the question statement may take a bit of deciphering, but all the information about the relationships is there.
A round wheel with a radius of 37 cm rolls at a constant speed of 3 revolutions per second. How far does the axle of the wheel move in 8 seconds?	**Closed** – you need to make sure you understand the more technical words and you need to know how speed, distance and time are related, but you don't need any extra data to get the answer.
Is it better for you to walk or cycle to university?	**Open** – for a start you need to define what 'better' might mean in this context.

ACTIVITY 2 What are you looking for?

	The solution will be …
If Tom has three times as many apples as Susan and Susan has a quarter as many as Joe, who has four, how many does Mary have if Mary has two more than Tom?	How many apples Mary has. The answer will be a number.
Design an efficient tin opener.	A diagram or description in words. To give a more precise description of the answer for this open problem you would need to work on the problem statement to define it more precisely. For instance, in what way should the tin opener be efficient?
Solve $2x + 3y = 1$ $x - 4y = 17$	Values for *x* and *y*. The answer will be two numbers.

ACTIVITY 3 Identifying givens

	I have been given …
If Tom has three times as many apples as Susan and Susan has a quarter as many as Joe, who has four, how many does Mary have if Mary has two more than Tom?	The number of apples Joe has ($J = 4$). The relationship between the number of apples Susan has and the number Joe has ($S = J/4$). The relationship between the number of apples Tom has and the number Susan has ($T = 3S$). The relationship between the number of apples Mary has and the number Tom has ($M = T + 2$).
Design an efficient tin opener is	Very little – I know what function a tin opener has, but I will need to find a definition for *efficient*. This a very open problem and open problems are characterised by having fewer 'givens' and therefore more that you must define yourself by research or thinking.
Solve $2x + 3y = 1$ $x - 4y = 17$	Two relationships between the value of x and the value of y.

ACTIVITY 4 Using your expert knowledge

Problem	Expert knowledge needed
1. Mary looked out of her window and saw some chickens and cows passing by. She counted all the legs of the chickens and cows and found that the total number of legs added up to 66. How many of each kind of animal passed by her window if the total number of animals is 24?	How many legs a chicken has. How many legs a cow has.
2. A car travels at 60 mph. How far does it travel in 30 minutes?	The meaning of mph. The relationship between speed, distance travelled and time.

ACTIVITY 5 Drawing a diagram

Problem	Diagram
1. Cooking the dinner	You might find a time line to be the best diagram for this one. You can put in the time for dinner and for dessert to be served and work backwards from there.
2. Leaning ladder	This is a more traditional type of diagram for a mechanics problem. You can start with the wall, the floor and the ladder. You may need some expert knowledge to put all the force arrows in.
3. Tea party	A Venn diagram with a circle for each type of food might help you to sort this one out.

ACTIVITY 6 Breaking into parts

1. You wish to build a dam across a river. How will you design your dam?

Broad initial questions
What is the budget?
What is the time scale for construction?
What is the dam for? It might be to control flooding, to generate hydroelectric power or it may have more than one purpose.

You might also have thought of the environmental impact, the geology and topology of the area, communications with the build site, availability of materials, annual rainfall, etc.

More specific questions
Where will you put the dam?
What shape should it be?
What dimensions should it have?

You may also have added questions about materials to use, number of workers required, equipment needed, etc.

Detailed questions
What constraints does the local topography place on the shape?
What load will be on the dam due to the water behind it?
Will it need to have a road on the top?

2. A cubic polynomial is given as $P(x) = ax^3 + 7x^2 + 2x–b$. When x = –1, $P(x) = 0$ and when $x = 1$, $P(x) = 8$. Sketch the graph for $P(x)$ showing clearly where it crosses both axes.

The steps could be:

Step 1: Use the remainder theorem to derive the system of equations that *a* and *b* must satisfy.
Step 2: Solve the equations from step 1 and by using the values of *a* and *b*, rewrite $P(x)$ with the values you find replacing the symbols *a* and *b*, in instead of *a* and *b*.
Step 3: By using the remainder theorem show that $x + 3$ is a factor of $P(x)$.
Step 4: Factorise $P(x)$ into linear factors and solve $P(x) = 0$.
Step 5: Find the value of $P(x)$ when $x = 0$.

You now have everything you need to sketch the graph.

ACTIVITY 7 Simplify to find the pattern

1. The factors of 360 add up to 1170. What is the sum of the reciprocals of the factors?

You are trying to find a value for

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{120}+\frac{1}{180}+\frac{1}{360}$$

What about trying the same thing with the factors of 12 or of 15 and looking for a pattern?

The answer you are looking for is 1170/360.

2. Find the sum of the first 25 odd numbers.

Start with 1, then 1 + 3, then 1 + 3 + 5, etc. What is the pattern?

The answer you are looking for is 625.

ACTIVITY 8 How big and how small?

Estimate the approximate magnitude of the following:

Item	Approximate size
Wavelength of ultraviolet light	400 nm
Land area of Great Britain	241590 km^2
Thickness of a grain of salt	1 mm
Diameter of the earth	107 m
Diameter of a golf ball	4 cm
Speed of an average person walking	4.5 kph
Distance from the earth to the sun	150×10^9 m
Diameter of a human hair	100×10^{-6} m

ACTIVITY 9 How big will the answer be?

Problem	Estimator	Approximate answer
983 × 256 × 198	1000 × 250 × 200 = 1 × 10^3 × 25 × 10 × 2 × 100	50 × 10^6
(2439 × 1220 × 56) ÷ 430	(2500 × 1200 × 50) ÷ 400 = (25 × 100 × 12 × 100 × 5 × 10) ÷ (4 × 100) = (25 × 3 × 100 × 5 × 10) ÷ (1)	375 × 10^3
5236 + 7805 + 2300 – 4534	= 5000 + 7000 + 2000 – 4500	10500
(34 387 × 54 567) + 585 734	= (34 000 × 54 000) + 580 000	1.8 × 10^9. Note that adding the final value makes little difference to the estimate as it is four orders of magnitude smaller than the product of the first two terms so it has no effect on the first two figures.
((385 730 × 348) – 57) ÷ 680	= ((380 000 × 350)) – 50 ÷ 700	190 × 10^3. Again – not worth subtracting 50 from something of the order of 10^6 when only looking for an estimate.

References

- Pólya, G. (1990) *How to solve it*, 2nd edn. Harmondsworth, Penguin Books.
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